Control Complexity in Fallback Voting*

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Abstract

We study the control complexity of fallback voting. Like manipulation and bribery, electoral control describes ways of changing the outcome of an election; unlike manipulation or bribery attempts, control actions—such as adding/deleting/partitioning either candidates or voters—modify the participative structure of an election. Via such actions one can try to either make a favorite candidate win ("constructive control") or prevent a despised candidate from winning ("destructive control"). Computational complexity can be used to protect elections from control attempts, i.e., proving an election system resistant to some type of control shows that the success of the corresponding control action, though not impossible, is computationally prohibitive.

We show that fallback voting, an election system combining approval with majority voting [BS09], is resistant to each of the common types of candidate control and to each common type of constructive control. Among natural election systems with a polynomial-time winner problem, only plurality and sincere-strategy preference-based approval voting (SP-AV) were previously known to be fully resistant to candidate control [BTT92, HHR07, ENR09], and only Copeland voting and SP-AV were previously known to be fully resistant to constructive control [FHHR09a, ENR09]. However, plurality has fewer resistances to voter control, Copeland voting has fewer resistances to destructive control, and SP-AV (which like fallback voting has 19 out of 22 proven control resistances) is arguably less natural a system than fallback voting.

1 Introduction

Voting is a way of aggregating individual preferences (or votes) to achieve a societal consensus on which among several alternatives (or candidates) to choose. This is a central method of decision-

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making not only in human societies but also in, e.g., multiagent systems where autonomous software agents may have differing individual preferences on a given number of alternatives. Voting has been studied intensely in areas as diverse as social choice theory and political science, economics, operations research, artificial intelligence, and other fields of computer science. Voting applications in computer science include the web-page ranking problem [DKNS01], similarity search [FKS03], planning [ER93], and recommender systems [GMHS99]. For such applications, it is important to understand the computational properties of election systems.

Various ways of tampering with the outcome of elections have been studied from a complexity-theoretic perspective, in particular the complexity of changing an election's outcome by *manipulation* [BTT89, BO91, CSL07, HH07, FHHR09c], *control* [BTT92, HHR07, FHHR09a, HHR09, ENR09, FHHR09c], and *bribery* [FHH09, FHHR09a], see also the surveys by Faliszewski et al. [FHHR09b] and Baumeister et al. [BEH$^+$09].

In control scenarios, an external actor—commonly referred to as the "chair"—seeks to either make a favorite candidate win (constructive control) or block a despised candidate's victory (destructive control) via actions that change the participative structure of the election. Such actions include adding, deleting, and partitioning either candidates or voters; the 22 commonly studied control actions, and their corresponding control problems, are described formally in Section 2.

We study the control complexity of fallback voting, an election system introduced by Brams and Sanver [BS09] as a way of combining approval and preference-based voting. We prove that fallback voting is resistant (i.e., the corresponding control problem is NP-hard) to each of the 14 common types of candidate control. In addition, we show that fallback voting is resistant to five types of voter control. In particular, fallback voting is resistant to each of the 11 common types of constructive control. Among natural election systems with a polynomial-time winner determination procedure, only plurality and sincere-strategy preference-based approval voting (SP-AV) were previously known to be fully resistant to candidate control [BTT92, HHR07, ENR09], and only Copeland voting and SP-AV were previously known to be fully resistant to constructive control [FHHR09a, ENR09]. However, SP-AV (as modified by [ENR09]) is arguably less natural a system than fallback voting,[1] and plurality has fewer resistances to voter control and Copeland voting has fewer resistances to destructive control than fallback voting.

This paper is organized as follows. In Section 2, we recall some notions from voting theory, define the commonly studied types of control, and explain Brams and Sanver's fallback voting procedure [BS06] in detail. Our results on the control complexity of fallback voting are presented in Section 3. Finally, Section 4 provides some conclusions and open questions.

[1] SP-AV is (a variant of) another hybrid system combining approval and preference-based voting that was also proposed by Brams and Sanver (see [BS06]). The reason we said SP-AV is less natural than fallback voting is that, in order to preserve "admissibility" of votes (as required by Brams and Sanver [BS06] to preclude trivial approval strategies), SP-AV (as modified by Erdélyi et al. [ENR09]) employs an additional rule to (re-)coerce admissibility (in particular, if in the course of a control action an originally admissible vote becomes inadmissible). This point has been discussed in detail by Baumeister et al. [BEH$^+$09]. In a nutshell, this rule, if applied, changes the approval strategies of the votes originally cast by the voters. The effect of this rule is that SP-AV can be seen as a hybrid between approval and plurality voting, and it indeed possesses each resistance either of these two systems has (and many of these resistance proofs are based on slightly modified constructions from the resistance proofs for either plurality or approval due to Hemaspaandra et al. [HHR07]). In contrast, here we study the original variant of fallback voting, as proposed by Brams and Sanver [BS09], in which votes, once cast, do not change.

2 Preliminaries

2.1 Elections and Electoral Control

An election is a pair (C,V), where C is a finite set of candidates and V is a collection of votes over C. How the votes are represented depends on the election system used. Many systems (such as majority, Condorcet, and Copeland voting as well as the class of scoring protocols including plurality, veto, and Borda voting; see, e.g., [BF02]) represent the voters' preferences as strict, linear orders over the candidates. In approval voting [BF78, BF83, Bra80], however, a vote over C is a so-called approval strategy, a subset of C containing all candidates approved of by this voter, whereas he or she disapproves of all other candidates. As is standard, we assume that collections V of votes over C are given as lists of ballots, i.e., if, say, five voters express the same preference (be it as a linear order or as an approval strategy or both) then this vote occurs five times in V.

An election system is a rule determining the winner(s) of a given election (C,V). For example, in *plurality-rule elections*, the winners are precisely those candidates in C who are ranked first place by the most voters. In *(simple) majority-rule elections*, the winner is that candidate in C (assuming one exists) who is ranked first place by a strict majority of the votes (i.e., by more than $\|V\|/2$ voters). In *approval voting*, every candidate scores one point for each approval by a voter, and whoever has the most points wins.

All our control problems come in two variants: The *constructive* case [BTT92] focuses on the chair seeking to make a favorite candidate win; the *destructive* case [HHR07] is concerned with the chair seeking to make a despised candidate not win. Due to space, we describe these control problems only very briefly. Detailed, formal problem descriptions, along with many motivating examples and scenarios where these types of control naturally occur in real-world elections, can be found, e.g., in [BTT92, HHR07, FHHR09a, HHR09, ENR09, BEH+09]. As an example, we state one such problem formally:

Name: Constructive Control by Adding a Limited Number of Candidates.

Given: An election $(C \cup D, V)$, $C \cap D = \emptyset$, a distinguished candidate $c \in C$, and a nonnegative integer k. (C is the set of originally qualified candidates and D is the set of spoiler candidates that may be added.)

Question: Does there exist a subset $D' \subseteq D$ such that $\|D'\| \leq k$ and c is the unique winner (under the election system at hand) of election $(C \cup D', V)$?

Constructive Control by Adding an Unlimited Number of Candidates is the same except there is no limit k on the number of spoiler candidates that may be added. The destructive variants of both problems are obtained by asking whether c is *not* a unique winner of $(C \cup D', V)$. The constructive control by deleting candidates problem is defined analogously except that we are given an election (C,V), a candidate $c \in C$, and a deletion limit k, and ask whether c can be made a unique winner by deleting up to k candidates from C. The destructive version of this problem is the same except that we now want to preclude c from being a unique winner (and we are not allowed to delete c).

Constructive Control by Partition (or Run-Off Partition) of Candidates takes as input an election (C,V) and a candidate $c \in C$ and asks whether c can be made a unique winner in a certain two-stage election consisting of one (or two) first-round subelection(s) and a final round. In both

variants, following [HHR07], we consider two tie-handling rules, TP ("ties promote") and TE ("ties eliminate"), that enter into force when more candidates than one are tied for winner in any of the first-round subelections. In the variant with run-off and under the TP rule, the question is whether C can be partitioned into C_1 and C_2 such that c is the unique winner of election $(W_1 \cup W_2, V)$, where W_i, $i \in \{1,2\}$, is the set of winners of subelection (C_i, V). In the variant without run-off (again under TP), the question is whether C can be partitioned into C_1 and C_2 such that c is the unique winner of election $(W_1 \cup C_2, V)$. In both cases, when the TE rule is used, none of multiple, tied first-round subelection winners is promoted to the final round (e.g., if we have a run-off and $\|W_2\| \geq 2$ then the final-round election collapses to (W_1, V); only a unique first-round subelection winner is promoted to the final round). It is obvious how to obtain the destructive variants of these eight problems formalizing control by candidate partition. Summing up, we now have defined 14 candidate control problems.

Constructive Control by Adding Voters is the problem of deciding, given an election $(C, V \cup V')$, $V \cap V' = \emptyset$, where V is a collection of registered voters and V' a pool of as yet unregistered voters that can be added, a candidate $c \in C$, and an addition limit k, whether there is a subset $V'' \subseteq V'$ of size at most k such that c is the unique winner of election $(C, V \cup V'')$. Constructive Control by Deleting Voters asks, given an election (C, V), a candidate $c \in C$, and a deletion limit k, whether it is possible to make c the unique winner by deleting up to k votes from V. For the TP rule, in Constructive Control by Partition of Voters the question is, given an election (C, V) and a candidate $c \in C$, whether V can be partitioned into V_1 and V_2 such that c is the unique winner of election $(W_1 \cup W_2, V)$, where W_i, $i \in \{1,2\}$, is the set of winners of subelection (C, V_i). It is obvious how this problem definition changes when the TE rule is used, and also how the destructive variants of these four voter control problems are obtained. Summing up, we now have defined eight voter control problems and thus a total of 22 control problems. These problems are due to [BTT92] and [HHR07] (see also [FHHR09a]), and their complexity has been studied for a variety of election systems, with the goal of using complexity as a barrier that makes attempts of changing the outcome of an election via control, although not impossible, at least computationally prohibitive.

We assume the reader is familiar with the standard complexity classes P (deterministic polynomial time) and NP (nondeterministic polynomial time) and with the notions of hardness and completeness. In particular, a problem B is said to be NP-hard if for each $A \in \mathrm{NP}$ it holds that A (polynomial-time many-one) reduces to B, where we say A reduces to B if there is a polynomial-time function φ such that for each input x, x is in A if and only if $\varphi(x)$ is in B. Every NP-hard problem in NP is said to be NP-complete. For more background on complexity theory, we refer to the textbooks [GJ79, Pap94, Rot05].

Given a control type Φ, some election systems have the advantageous property that it is never possible for the chair to reach his or her goal of exerting control of type Φ. In such a case, the system is said to be *immune to* Φ; otherwise, the system is said to be *susceptible to* Φ. We say an election system $\mathscr{E}$ is *resistant to* Φ if $\mathscr{E}$ is susceptible to Φ and the control problem corresponding to Φ is NP-hard. If, however, $\mathscr{E}$ is susceptible to Φ and the control problem corresponding to Φ is in P, we say $\mathscr{E}$ is *vulnerable to* Φ.

2.2 Fallback Voting

Brams and Sanver proposed two hybrid systems that combine approval voting and systems based on linear preferences, *sincere-strategy preference-based approval voting* [BS06] and *fallback voting* (FV) [BS09]. Erdélyi et al. [ENR09] studied the control complexity of SP-AV, as they dubbed their variant of the election system originally proposed by [BS06]. As explained in the last paragraph of the introduction, SP-AV combines, in a certain sense, approval with plurality voting. In this paper, we investigate the election system FV with respect to electoral control. FV can be thought of as combining, in a certain sense, approval with majority voting. Unlike in SP-AV, in FV only the approved candidates are ranked by a tie-free linear order.

Definition 2.1 ([BS09])**.** *Let (C,V) be an election. Every voter $v \in V$ provides a subset $S_v \subseteq C$ indicating that v approves of all candidates in S_v and disapproves of all candidates in $C - S_v$. S_v is called v's* approval strategy. *In addition, each voter $v \in V$ provides a tie-free linear ordering of all candidates in S_v.*

If $S_v = \{c_1, c_2, \dots, c_k\}$ and v ranks the candidates in S_v by $c_1 > c_2 > \cdots > c_k$, where c_1 is v's most preferred candidate, c_2 is v's second most preferred candidate, etc., and c_k is v's least preferred candidate (among the candidates v approves of), we denote the vote v by

$$c_1 \ \ c_2 \ \ \cdots \ \ c_k \ \mid \ C - S_v,$$

where the approved candidates to the left of the approval line are ranked and the disapproved candidates in $C - S_v$ are written as an (unordered) set to the right of the approval line. For each $c \in C$, let

$$score_{(C,V)}(c) = \|\{v \in V \mid c \in S_v\}\|$$

denote the number of c's approvals in (C,V), and let $score^i_{(C,V)}(c)$ be the level i score of c in (C,V), *which is the number of c's approvals when ranked between the (inclusively) first and i^{th} position. Winner determination in FV is based on determining level i FV winners as follows:*

1. *On the first level, only the highest ranked approved candidates are considered in each voters' approval strategy. If a candidate $c \in C$ has a strict majority on this level (i.e., $score^1_{(C,V)}(c) > \|V\|/2$), then c is the* (unique) level 1 FV winner of the election.

2. *On the second level, if there is no level 1 winner, the two highest ranked approved candidates (if they exist) are considered in each voters' approval strategy. If there is exactly one candidate $c \in C$ who has a strict majority on this level (i.e., $score^2_{(C,V)}(c) > \|V\|/2$), then c is the* (unique) level 2 FV winner of the election. *If there are at least two such candidates, then every candidate with the highest level 2 score is a* level 2 FV winner of the election.

3. *If there is no level 1 or level 2 FV winner, we in this way continue level by level until there is at least one candidate who was approved by a strict majority on the current level, say level i. If there is only one such candidate, he or she is the* (unique) level i FV winner of the election. *If there are at least two such candidates, then every candidate with the highest level i score is a* level i FV winner of the election.

Control by	FV		SP-AV		AV	
	Const.	Dest.	Const.	Dest.	Const.	Dest.
Adding an Unlimited Number of Candidates	**R**	**R**	R	R	I	V
Adding a Limited Number of Candidates	**R**	**R**	R	R	I	V
Deleting Candidates	**R**	**R**	R	R	V	I
Partition of Candidates	**TE: R** **TP: R**	**TE: R** **TP: R**	TE: R TP: R	TE: R TP: R	TE: V TP: I	TE: I TP: I
Run-off Partition of Candidates	**TE: R** **TP: R**	**TE: R** **TP: R**	TE: R TP: R	TE: R TP: R	TE: V TP: I	TE: I TP: I
Adding Voters	**R**	**V**	R	V	R	V
Deleting Voters	**R**	**V**	R	V	R	V
Partition of Voters	**TE: R** **TP: R**	**TE: S** **TP: R**	TE: R TP: R	TE: V TP: R	TE: R TP: R	TE: V TP: V

Table 1: Overview of results. Key: I means immune, S means susceptible, R means resistant, V means vulnerable, TE means ties eliminate, and TP means ties promote. Results new to this paper are in boldface. Results for approval voting are due to [HHR07] and results for SP-AV are due to [ENR09].

4. *For each $c \in C$, if c is a level i FV winner of (C,V) for some (smallest) $i \leq \|C\|$ then c is said to be an* FV winner of (C,V). *Otherwise (i.e., if for no $i \leq \|C\|$ there is a level i FV winner), every candidate with the highest score$_{(C,V)}(c)$ is an* FV winner of (C,V).

Note that if there exists a level 1 FV winner, he or she is always the election's unique FV winner. In contrast to SP-AV (where *admissibility* of the votes—$\emptyset \neq S_v \neq C$ for each $v \in V$—is coerced by moving the approval line whenever necessary, see [ENR09]), in fallback voting no changes are made to the ballots, regardless of the control action taken. That is, we don't require votes to be admissible, i.e., both empty ($S_v = \emptyset$) and full ($S_v = C$) approval strategies are allowed. By definition, votes in an FV election are always *sincere* (i.e., if a voter v approves of a candidate c then the voter also approves of all candidates ranked higher than c). In contrast, sincerity has to be enforced in SP-AV by declaring insincere votes void.

3 Results

3.1 Overview

Theorem 3.1 and Table 1 show the results on the control complexity of fallback voting.

Theorem 3.1. *Fallback voting is resistant, vulnerable, and susceptible to the* 22 *types of control defined in Section 2 as shown in Table 1.*

3.2 Susceptibility

Among the 22 control types we consider, approval voting has nine immunities [HHR07], see Table 1. Some of these immunities immediately follow from the unique version of the Weak Axiom

of Revealed Preference (Unique-WARP, for short), which says that if a candidate c is the unique winner of an election (C,V) then c is also the unique winner of each election (C',V) such that $C' \subset C$ and $c \in C'$ (where, for convenience, we use the same symbol V but view the preferences in V as being restricted to the candidates in C'; this convention is also adopted when we speak of subelections in the context of candidate-control problems).

Unlike approval voting but just as SP-AV, fallback voting does not satisfy Unique-WARP.

Proposition 3.2. *Fallback voting does not satisfy Unique-WARP.*

Proof. Consider the election (C,V) with candidate set $C = \{a,b,c,d\}$ and voter collection $V = \{v_1, v_2, \dots, v_6\}$:

$$\begin{array}{r|l} & (C,V) \\ \hline v_1 = v_2 = v_3: & a \ \ c \quad\ \mid \{b,d\} \\ v_4 = v_5: & b \ \ d \ \ c \mid \ \{a\} \\ v_6: & d \ \ a \ \ c \mid \ \{b\} \end{array}$$

There is no level 1 FV winner, and the unique level 2 FV winner of the election (C,V) is candidate a with $score^2_{(C,V)}(a) = 4$. By removing candidate b from the election, we get the subelection (C',V) with $C' = \{a,c,d\}$. (Recall that, after removing b, V is viewed as restricted to C'; e.g., voter v_4 in V is now changed to $d \ \ c \mid \{a\}$.) There is again no level 1 FV winner in (C',V). However, there are two candidates on the second level with a strict majority, namely candidates a and c. Since $score^2_{(C',V)}(c) = 5$ is greater than $score^2_{(C',V)}(a) = 4$, the unique level 2 FV winner of the subelection (C',V) is candidate c. Thus, FV does not satisfy Unique-WARP. ❑

Indeed, as we will now show, fallback voting is susceptible to each of our 22 control types. Our proofs make use of the results of [HHR07] that provide general proofs of and links between certain susceptibility cases. For the sake of self-containment, we state their results, as Theorems A.1, A.2, and A.3, in the appendix.

We start with susceptibility to candidate control.

Lemma 3.3. *Fallback voting is susceptible to constructive and destructive control by adding candidates (in both the "limited" and the "unlimited" case), by deleting candidates, and by partition of candidates (with or without run-off and for each in both model TE and model TP).*

Proof. From Theorem A.1 and the fact that FV is a voiced voting system,[2] it follows that FV is susceptible to constructive control by deleting candidates, and to destructive control by adding candidates (in both the "limited" and the "unlimited" case).

Now, consider the election (C,V) given in the proof of Proposition 3.2. The unique FV winner of the election is candidate a. Partition C into $C_1 = \{a,c,d\}$ and $C_2 = \{b\}$. The unique FV winner of subelection (C_1,V) is candidate c, as shown in the proof of Proposition 3.2. In both partition and run-off partition of candidates and for each in both tie-handling models, TE and TP, candidate b runs against candidate c in the final stage of the election. The unique FV winner is in each case candidate

[2]An election system is said to be *voiced* if the single candidate in any one-candidate election always wins.

c. Thus, FV is susceptible to destructive control by partition of candidates (with or without run-off and for each in both model TE and model TP).

By Theorem A.3, FV is also susceptible to destructive control by deleting candidates. By Theorem A.2, FV is also susceptible to constructive control by adding candidates (in both the "limited" and the "unlimited" case).

Now, changing the roles of a and c makes c our distinguished candidate. In election (C,V), c loses against candidate a. By partitioning the candidates as described above, c becomes the unique FV winner of the election. Thus, FV is susceptible to constructive control by partition of candidates (with or without run-off and for each in both tie-handling models, TE and TP). ❑

We now turn to susceptibility to voter control.

Lemma 3.4. *Fallback voting is susceptible to constructive and destructive control by adding voters, by deleting voters, and by partition of voters (in both model TE and model TP).*

Proof. Consider the election (C,V), where $C = \{a,b,c,d\}$ is the set of candidates and $V = \{v_1,v_2,v_3,v_4\}$ is the collection of voters with the following preferences:

$$\begin{array}{r|l}
 & (C,V) \\
\hline
v_1: & a\ c \quad | \ \{b,d\} \\
v_2: & d\ c \quad | \ \{a,b\} \\
v_3: & b\ a\ c \ | \ \{d\} \\
v_4: & b\ a \quad | \ \{c,d\}
\end{array}$$

We partition V into $V_1 = \{v_1,v_2\}$ and $V_2 = \{v_3,v_4\}$. Thus we split (C,V) into two subelections:

$$\begin{array}{r|l c l}
 & (C,V_1) & \text{and} & (C,V_2) \\
\hline
v_1: & a\ c \ | \ \{b,d\} & & \\
v_2: & d\ c \ | \ \{a,b\} & & \\
v_3: & & & b\ a\ c \ | \ \{d\} \\
v_4: & & & b\ a \quad | \ \{c,d\}
\end{array}$$

Clearly, candidate a is the unique level 2 FV winner of (C,V). However, c is the unique level 2 FV winner of (C,V_1) and b is the unique level 1 FV winner of (C,V_2), and so a is not promoted to the final stage. Thus, FV is susceptible to destructive control by partition of voters in both tie-handling models, TE and TP.

By Theorem A.1 and the fact that FV is a voiced voting system, FV is susceptible to destructive control by deleting voters. By Theorem A.2, FV is also susceptible to constructive control by adding voters.

By changing the roles of a and c again, we can see that FV is susceptible to constructive control by partition of voters in both model TE and model TP. By Theorem A.3, FV is also susceptible to constructive control by deleting voters. Finally, again by Theorem A.2, FV is susceptible to destructive control by adding voters. ❑

3.3 Candidate Control

All resistance results in this section follow via Lemma 3.3, showing susceptibility, and a reduction from the well-known NP-complete problem Hitting Set [GJ79], showing NP-hardness of the corresponding control problem. Hitting Set is defined as follows:

Name: Hitting Set.
Instance: A set $B = \{b_1, b_2, \ldots, b_m\}$, a collection $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ of nonempty subsets $S_i \subseteq B$, and a positive integer $k \leq m$.
Question: Does $\mathscr{S}$ have a hitting set of size at most k, i.e., is there a set $B' \subseteq B$ with $\|B'\| \leq k$ such that for each i, $S_i \cap B' \neq \emptyset$?

We now show that fallback voting is resistant to all types of constructive and destructive candidate control defined in Section 2. To this end, we present a general construction that will be applied (in Theorems 3.7, 3.8, and 3.9 below) to all these control scenarios except constructive control by deleting candidates (which will be handled via a different construction in Theorem 3.10).

Construction 3.5. *Let* $(B, \mathscr{S}, k)$ *be a given instance of* Hitting Set, *where* $B = \{b_1, b_2, \ldots, b_m\}$ *is a set,* $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ *is a collection of nonempty subsets* $S_i \subseteq B$*, and* k *is a positive integer. Without loss of generality, we may assume that* $n > 1$ *(since* Hitting Set *is trivially solvable for* $n \leq 1$*) and that* $k < m$ *(since* B *is always a hitting set of size* k *if* $m = k$ *when* $\mathscr{S}$ *contains only nonempty sets).*

Define the election (C, V)*, where* $C = B \cup \{c, d, w\}$ *is the candidate set and where* V *consists of the following* $6n(k+1) + 4m + 11$ *voters:*

1. *There are* $2m+1$ *voters of the form:*

$$c \mid B \cup \{d, w\}.$$

2. *There are* $2n + 2k(n-1) + 3$ *voters of the form:*

$$c \;\; w \mid B \cup \{d\}.$$

3. *There are* $2n(k+1) + 5$ *voters of the form:*

$$w \;\; c \mid B \cup \{d\}.$$

4. *For each* i*,* $1 \leq i \leq n$*, there are* $2(k+1)$ *voters of the form:*[3]

$$d \;\; S_i \;\; c \mid (B - S_i) \cup \{w\}.$$

[3]As a notation, when a vote contains a set of approved candidates, such as $c \;\; D \;\; a \mid C - (D \cup \{a, c\})$ for a subset $D \subseteq C$ of the candidates, this is a shorthand for $c \;\; d_1 \;\; \cdots \;\; d_\ell \;\; a \mid C - (D \cup \{a, c\})$, where the elements of $D = \{d_1, \ldots, d_\ell\}$ are ranked with respect to a (tacitly assumed) fixed ordering of all candidates in C.

5. *For each j, $1 \leq j \leq m$, there are two voters of the form:*

$$d \;\; b_j \;\; w \;\mid\; (B - \{b_j\}) \cup \{c\}.$$

6. *There are $2(k+1)$ voters of the form:*

$$d \;\; w \;\; c \;\mid\; B.$$

It is easy to see that there is no level 1 FV winner in election $(\{c,d,w\},V)$ and that we have the following level 2 scores in this election:

$$\begin{aligned}
score^2_{(\{c,d,w\},V)}(c) &= 2(m-k)+6n(k+1)+9, \\
score^2_{(\{c,d,w\},V)}(d) &= 2n(k+1)+2(m+k+1), \text{ and} \\
score^2_{(\{c,d,w\},V)}(w) &= 4n(k+1)+2m+10.
\end{aligned}$$

Thus, c is the unique level 2 FV winner of $(\{c,d,w\},V)$.

The proofs of Theorems 3.7, 3.8, and 3.9 below will make use of the following lemma.

Lemma 3.6. *Consider the election (C,V) constructed according to Construction 3.5 from a* Hitting Set *instance $(B,\mathscr{S},k)$.*

1. *If $\mathscr{S}$ has a hitting set B' of size k, then w is the unique FV winner of election $(B' \cup \{c,d,w\},V)$.*
2. *Let $D \subseteq B \cup \{d,w\}$. If c is not a unique FV winner of election $(D \cup \{c\},V)$, then there exists a set $B' \subseteq B$ such that*
   (a) *$D = B' \cup \{d,w\}$,*
   (b) *w is the unique level 2 FV winner of election $(B' \cup \{c,d,w\},V)$, and*
   (c) *B' is a hitting set for $\mathscr{S}$ of size at most k.*

Proof. For the first part, suppose that B' is a hitting set for $\mathscr{S}$ of size k. Then there is no level 1 FV winner in election $(B' \cup \{c,d,w\},V)$, and we have the following level 2 scores:

$$\begin{aligned}
score^2_{(B' \cup \{c,d,w\},V)}(c) &= 4n(k+1)+2(m-k)+9, \\
score^2_{(B' \cup \{c,d,w\},V)}(d) &= 2n(k+1)+2(m+k+1), \\
score^2_{(B' \cup \{c,d,w\},V)}(w) &= 4n(k+1)+2(m-k)+10, \\
score^2_{(B' \cup \{c,d,w\},V)}(b_j) &\leq 2n(k+1)+2 \quad \text{for all } b_j \in B'.
\end{aligned}$$

It follows that w is the unique level 2 FV winner of election $(B' \cup \{c,d,w\},V)$.

For the second part, let $D \subseteq B \cup \{d,w\}$. Suppose c is not a unique FV winner of election $(D \cup \{c\},V)$.

(2a) Other than c, only w is approved by a strict majority of voters and only w can tie or beat c by the number of approvals in $(D \cup \{c\}, V)$. Thus, since c is not a unique FV winner of election $(D \cup \{c\}, V)$, w is clearly in D. In $(D \cup \{c\}, V)$, candidate w has no level 1 strict majority, and candidate c has already on level 2 a strict majority. Thus, w must tie or beat c on level 2. For a contradiction, suppose $d \notin D$. Then

$$score^2_{(D\cup\{c\},V)}(c) \geq 4n(k+1)+2m+11.$$

The overall score of w is

$$score_{(D\cup\{c\},V)}(w) = 4n(k+1)+2m+10,$$

which contradicts our assumption, that w ties or beats c on level 2. Thus, $D = B' \cup \{d, w\}$, where $B' \subseteq B$.

(2b) This part follows immediately from part (2a).

(2c) Let ℓ be the number of sets in $\mathscr{S}$ not hit by B'. We have that

$$\begin{aligned} score^2_{(B'\cup\{c,d,w\},V)}(w) &= 4n(k+1)+10+2(m-\|B'\|), \\ score^2_{(B'\cup\{c,d,w\},V)}(c) &= 2(m-k)+4n(k+1)+9+2(k+1)\ell. \end{aligned}$$

From part (2a) we know that

$$score^2_{(B'\cup\{c,d,w\},V)}(w) \geq score^2_{(B'\cup\{c,d,w\},V)}(c),$$

so

$$4n(k+1)+10+2(m-\|B'\|) \quad \geq \quad 2(m-k)+4n(k+1)+9+2(k+1)\ell.$$

The above inequality implies

$$1 > \frac{1}{2} \geq \|B'\| - k + (k+1)\ell \geq 0,$$

so $\|B'\| - k + (k+1)\ell = 0$. Thus $\ell = 0$, and it follows that B' is a hitting set for $\mathscr{S}$ of size at most k.

This completes the proof of Lemma 3.6. ❑

Theorem 3.7. *Fallback voting is resistant to constructive and destructive control by adding candidates (both in the limited and the unlimited version of the problem).*

Proof. Susceptibility holds by Lemma 3.3. NP-hardness follows immediately from Lemma 3.6, via mapping the Hitting Set instance $(B, \mathscr{S}, k)$ to the instance

1. $((\{c,d,w\} \cup B, V), w, k)$ of Constructive Control by Adding a Limited Number of Candidates,

2. $((\{c,d,w\} \cup B, V), c, k)$ of Destructive Control by Adding a Limited Number of Candidates,

3. $((\{c,d,w\} \cup B, V), w)$ of Constructive Control by Adding an Unlimited Number of Candidates, and

4. $((\{c,d,w\} \cup B, V), c)$ of Destructive Control by Adding an Unlimited Number of Candidates.

where in each case c, d, and w are the qualified candidates and B is the set of spoiler candidates. ❑

Theorem 3.8. *Fallback voting is resistant to destructive control by deleting candidates.*

Proof. Susceptibility holds by Lemma 3.3. To show the problem NP-hard, let (C,V) be the election resulting from a Hitting Set instance $(B, \mathscr{S}, k)$ according to Construction 3.5, and let c be the distinguished candidate.

We claim that $\mathscr{S}$ has a hitting set of size at most k if and only if c can be prevented from being a unique FV winner by deleting at most $m-k$ candidates.

From left to right: Suppose $\mathscr{S}$ has a hitting set B' of size k. Delete the $m-k$ candidates $B - B'$. Now, both candidates c and w have a strict majority on level 2, but

$$\begin{aligned} score^2_{(\{c,d,w\} \cup B', V)}(c) &= 4n(k+1) + 2(m-k) + 9, \\ score^2_{(\{c,d,w\} \cup B', V)}(w) &= 4n(k+1) + 2(m-k) + 10, \end{aligned}$$

so w is the unique level 2 FV winner of this election.

From right to left: Suppose that c can be prevented from being a unique FV winner by deleting at most $m-k$ candidates. Let $D' \subseteq B \cup \{d,w\}$ be the set of deleted candidates (so $c \notin D'$) and $D = (C - D') - \{c\}$. It follows immediately from Lemma 3.6 that $D = B' \cup \{d,w\}$, where B' is a hitting set for $\mathscr{S}$ of size at most k. ❑

Theorem 3.9. *Fallback voting is resistant to constructive and destructive control by partition of candidates and by run-off partition of candidates (for each in both tie-handling models, TE and TP).*

Proof. Susceptibility holds by Lemma 3.3, so it remains to show NP-hardness. For the constructive cases, map the given Hitting Set instance $(B, \mathscr{S}, k)$ to the election (C,V) from Construction 3.5 with distinguished candidate w.

We claim that $\mathscr{S}$ has a hitting set of size at most k if and only if w can be made a unique FV winner by exerting control via any of our four control scenarios (partition of candidates with or without run-off, and for each in either tie-handling model, TE and TP).

From left to right: Suppose $\mathscr{S}$ has a hitting set $B' \subseteq B$ of size k. Partition the set of candidates into the two subsets $C_1 = B' \cup \{c,d,w\}$ and $C_2 = C - C_1$. According to Lemma 3.6, w is the unique level 2 FV winner of subelection $(C_1, V) = (B' \cup \{c,d,w\}, V)$. Note that w's score in the final stage is at least $2(m-k) + 4n(k+1) + 9$. Since (no matter whether we have a run-off or not, and regardless of the tie-handling rule used) the opponents of w in the final stage (if there are any opponents at all) each are candidates from B whose score is at most $2n(k+1)+2$, w is the only candidate in the final stage with a strict majority of approvals. Thus, w is the unique FV winner of the resulting election.

From right to left: Suppose w can be made a unique FV winner via any of our four control scenarios. Since c is not an FV winner of the election, there is a subset $D \subseteq B \cup \{d, w\}$ of candidates such that c is not a unique FV winner of the election $(D \cup \{c\}, V)$. By Lemma 3.6, there exists a hitting set for $\mathscr{S}$ of size at most k.

For the four destructive cases, we simply change the roles of c and w in the above argument. ❑

Construction 3.5 does not work for constructive control by deleting candidates in fallback voting. By deleting c the chair could make w a unique FV winner, regardless of whether or not $\mathscr{S}$ has a hitting set of size k. The following theorem provides a different construction that shows resistance in this case as well.

Theorem 3.10. *Fallback voting is resistant to constructive control by deleting candidates.*

Proof. Susceptibility holds by Lemma 3.3. To show NP-hardness, let $(B, \mathscr{S}, k)$ be a Hitting Set instance with $B = \{b_1, b_2, \ldots, b_m\}$ a set, $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ a collection of nonempty subsets $S_i \subseteq B$, and $k \leq m$ a positive integer. Define the election (C, V) with candidate set

$$C = B \cup C' \cup D \cup E \cup \{w\},$$

where $C' = \{c_1, c_2, \ldots, c_{k+1}\}$, $D = \{d_1, d_2, \ldots, d_s\}$, $E = \{e_1, e_2, \ldots, e_n\}$, w is the distinguished candidate, and the number of candidates in D is $s = \sum_{i=1}^{n} s_i$ with $s_i = n + k - \|S_i\|$, so $s = n^2 + kn - \sum_{i=1}^{n} \|S_i\|$. Define V to be the following collection of $2(n+k+1)+1$ voters:

1. For each i, $1 \leq i \leq n$, there is one voter of the form:

   $$S_i \;\; D_i \;\; w \;\mid\; (B - S_i) \cup (D - D_i) \cup C' \cup E,$$

   where $D_i = \{d_{1+\sum_{j=1}^{i-1} s_j}, \ldots, d_{\sum_{j=1}^{i} s_j}\}$.

2. For each j, $1 \leq j \leq k+1$, there is one voter of the form:

   $$E \;\; (C' - \{c_j\}) \;\; c_j \;\mid\; B \cup D \cup \{w\}.$$

3. There are $k+1$ voters of the form:

   $$w \;\mid\; B \cup C' \cup D \cup E.$$

4. There are n voters of the form:

   $$C' \;\mid\; B \cup D \cup E \cup \{w\}.$$

5. There is one voter of the form:

   $$C' \;\; w \;\mid\; B \cup D \cup E.$$

Note that there is no unique FV winner in the above election; the candidates in C' and w are all level $n+k+1$ FV winners.

We claim that $\mathscr{S}$ has a hitting set of size k if and only if w can be made a unique FV winner by deleting at most k candidates.

From left to right: Suppose $\mathscr{S}$ has a hitting set B' of size k. Delete the corresponding candidates. Now, w is the unique level $(n+k)$ FV winner of the resulting election.

From right to left: Suppose w can be made a unique FV winner by deleting at most k candidates. Since $k+1$ candidates other than w have a strict majority on level $n+k+1$ in election (C,V), after deleting at most k candidates, there is still at least one candidate other than w with a strict majority of approvals on level $n+k+1$. However, since w was made a unique FV winner by deleting at most k candidates, w must be the unique FV winner on a level lower than or equal to $n+k$. This is possible only if in all n votes of the first voter group w moves forward by at least one position. This, however, is possible only if $\mathscr{S}$ has a hitting set B' of size k. ❑

3.4 Voter Control

All vulnerability results in this section follow via Lemma 3.4, showing susceptibility, and a polynomial-time algorithm for the corresponding control problem. All resistance results in this section follow via Lemma 3.4, showing susceptibility, and a reduction from either Hitting Set or the well-known NP-complete problem Exact Cover by Three-Sets [GJ79], which is defined as follows:

Name: Exact Cover by Three-Sets (X3C).
Instance: A set $B=\{b_1,b_2,\dots,b_{3m}\}$, $m\geq 1$, and a collection $\mathscr{S}=\{S_1,S_2,\dots,S_n\}$ of subsets $S_i\subseteq B$ with $\|S_i\|=3$ for each i, $1\leq i\leq n$.
Question: Is there a subcollection $\mathscr{S}'\subseteq\mathscr{S}$ such that each element of B occurs in exactly one set in $\mathscr{S}'$?

Our first result for voter control says that fallback voting is resistant to constructive control by adding voters and to constructive control by deleting voters.

Theorem 3.11. *Fallback voting is resistant to constructive control by adding voters and by deleting voters.*

Proof. Susceptibility holds by Lemma 3.4 in both cases. We first prove NP-hardness—and thus resistance—of Constructive Control by Adding Voters. Let $(B,\mathscr{S})$ be an X3C instance, where $B=\{b_1,b_2,\dots,b_{3m}\}$ is a set with $m>1$ and $\mathscr{S}=\{S_1,S_2,\dots,S_n\}$ is a collection of subsets $S_i\subseteq B$ with $\|S_i\|=3$ for each i, $1\leq i\leq n$. (Note that X3C is trivial to solve for $m=1$.)

Define the election $(C,V\cup V')$, where $C=B\cup\{w\}\cup D$ with $D=\{d_1,\dots,d_{n(3m-4)}\}$ is the set of candidates, w is the distinguished candidate, and $V\cup V'$ is the following collection of $n+m-2$ voters:[4]

[4]This construction—just as that for SP-AV [ENR09]—is based on the corresponding construction for approval voting [HHR07].

1. V is the collection of $m-2$ registered voters of the form:

$$B \mid D \cup \{w\}.$$

2. V' is the collection of unregistered voters, where for each i, $1 \leq i \leq n$, there is one voter of the form:

$$D_i \;\; S_i \;\; w \mid (B - S_i) \cup (D - D_i),$$

where $D_i = \{d_{(i-1)(3m-4)+1}, \ldots, d_{i(3m-4)}\}$.

Since w has no approvals in (C,V), w is not a unique FV winner in (C,V).

We claim that $\mathscr{S}$ has an exact cover for B if and only if w can be made a unique FV winner by adding at most m voters from V'.

From left to right: Suppose $\mathscr{S}$ contains an exact cover for B. Let V'' contain the corresponding voters from V' (i.e., voters of the form $D_i \;\; S_i \;\; w \mid (B - S_i) \cup (D - D_i)$ for each S_i in the exact cover) and add V'' to the election. It follows that $score_{(C,V \cup V'')}(d_j) = 1$ for all $d_j \in D$, $score_{(C,V \cup V'')}(b_j) = m-1$ for all $b_j \in B$, and $score_{(C,V \cup V'')}(w) = m$. Thus, only w has a strict majority of approvals and so is the unique FV winner of the election.

From right to left: Let $V'' \subseteq V'$ be such that $\|V''\| \leq m$ and w is the unique winner of election $(C, V \cup V'')$. Since w must in particular beat every $b_j \in B$, it follows that $\|V''\| = m$ and each $b_j \in B$ can gain only one additional point. Thus, the m voters in V'' correspond to an exact cover for B.

Next, we show that FV is resistant to constructive control by deleting voters. Let $(B, \mathscr{S})$ be an X3C instance as above. Define the election (C,V), where $C = B \cup \{c,w\} \cup D$ is the set of candidates with $D = \{d_1, d_2, \ldots, d_{3nm}\}$, w is the distinguished candidate, and V is the following collection of $2n+m-1$ voters:

1. For each i, $1 \leq i \leq n$, there is one voter of the form:

$$S_i \;\; c \mid (B - S_i) \cup D \cup \{w\}.$$

2. For each i, $1 \leq i \leq n$, there is one voter of the form:

$$B_i \;\; D_i \;\; w \mid (B - B_i) \cup \{c\} \cup (D - D_i),$$

where, letting $\ell_j = \|\{S_i \in \mathscr{S} \mid b_j \in S_i\}\|$ for each j, $1 \leq j \leq 3m$, $B_i = \{b_j \in B \mid i \leq n - \ell_j\}$ and $D_i = \{d_{(i-1)3m+1}, \ldots, d_{3im - \|B_i\|}\}$.

Note that $D_i = \emptyset$ if $\|B_i\| = 3m$. Note also that w is always ranked on the $(3m+1)^{st}$ place.

3. There are $m-1$ voters of the form:

$$c \mid B \cup D \cup \{w\}.$$

Note that $score(w) = score(b_i) = n$ for all i, $1 \leq i \leq 3m$, $score(d_j) = 1$ or $score(d_j) = 0$ for all $d_j \in D$, and $score(c) = n+m-1$, and since only c has a strict majority (which is reached on level 4), c is the unique level 4 FV winner of the election.

We claim that $\mathscr{S}$ has an exact cover for B if and only if w can be made a unique FV winner by deleting at most m voters.

From left to right: Suppose $\mathscr{S}$ contains an exact cover for B. By deleting the corresponding voters from the first voter group, we have the following scores: $score(w) = n$, $score(b_i) = score(c) = n-1$ for all i, $1 \leq i \leq 3m$, and $score(d_j) = 1$ or $score(d_j) = 0$ for all $d_j \in D$. Since there are now $2n-1$ voters in the election, only candidate w has a strict majority, so w is the unique FV winner of the election.

From right to left: Suppose w can be made a unique FV winner by deleting at most m voters. Since w's approvals are all on the $(3m+1)^{st}$ position, neither c nor any of the b_i can have a strict majority on any of the previous levels. In particular, candidate c must have lost exactly m points after deletion, and this is possible only if the m deleted voters are all from the first or third voter group. On the other hand, each $b_i \in B$ must have lost at least one point after deletion, and this is possible only if exactly m voters were deleted from the first voter group. These m voters correspond to an exact cover for B. ❑

In contrast to the constructive voter-control cases of Theorem 3.11, fallback voting is vulnerable to destructive control by adding voters and to destructive control by deleting voters. In fact, the proof of Theorem 3.12 shows something slightly stronger: FV is what [HHR07] call "certifiably vulnerable" to these two destructive voter-control cases, i.e., the algorithm we present in this proof for destructive control by adding voters even computes a successful control action if one exists (instead of only solving the corresponding decision problem).[5]

Theorem 3.12. *Fallback voting is vulnerable to destructive control by adding voters and by deleting voters.*

Proof. Susceptibility holds by Lemma 3.4 in both cases. We present a polynomial-time algorithm for solving the destructive control by adding voters case. We will make use of the following notation. Given an election (C,V), let $maj(V) = \lfloor \|V\|/2 \rfloor + 1$ and define the deficit of candidate $d \in C$ for reaching a strict majority in (C,V) on level i, $1 \leq i \leq \|C\|$, by

$$def^i_{(C,V)}(d) = maj(V) - score^i_{(C,V)}(d).$$

The input to our algorithm is an election $(C, V \cup V')$ (where C is the set of candidates, V is the collection of registered voters, and V' is the collection of unregistered voters), a distinguished candidate $c \in C$, and an integer ℓ (the number of voters allowed to be added). The algorithm either outputs a subset $V'' \subseteq V'$, $\|V''\| \leq \ell$, that describes a successful control action (if any exists), or indicates that control is impossible for this input.

We give a high-level description of the algorithm. We assume that c is initially the unique FV winner of election (C,V); otherwise, the algorithm simply outputs $V'' = \emptyset$ and halts, since there is no need to add any voters from V'.

Let $n = \max_{v \in V \cup V'} \|S_v\|$. Clearly, $n \leq \|C\|$. The algorithm proceeds in at most $n+1$ stages, where the last stage is the *approval stage* and checks whether c can be dethroned as a unique FV

[5] And the same holds for the algorithm showing that FV is vulnerable to destructive control by deleting voters, which is not presented here due to space.

winner by approval score via adding at most ℓ voters from V', and all preceding stages are *majority stages* that check whether a candidate $d \neq c$ can tie or beat c on level i via adding at most ℓ voters from V'. Since the first majority stage is slightly different from the subsequent majority stages, we describe both cases separately.

Majority Stage 1: For each candidate $d \in C - \{c\}$, check whether d can tie or beat c on the first level via adding at most ℓ voters from V'. To this end, first check whether

$$def^1_{(C,V)}(d) \leq \frac{\ell}{2}; \tag{3.1}$$

$$score^1_{(C,V)}(d) \geq score^1_{(C,V)}(c) - \ell. \tag{3.2}$$

If (3.1) or (3.2) fails to hold, this d is hopeless, so go to the next candidate (or to the next stage if all other candidates have already been checked in this stage). If (3.1) and (3.2) hold, find a set $V'_d \subseteq V'$ of largest cardinality such that $\|V'_d\| \leq \ell$ and all voters in V'_d approve of d on the first level but disapprove of c on the first level. Check whether

$$score^1_{(C,V\cup V'_d)}(d) \geq score^1_{(C,V\cup V'_d)}(c). \tag{3.3}$$

If (3.3) fails to hold, this d is hopeless, so go to the next candidate (or to the next stage if all other candidates have already been checked in this stage). If (3.3) holds, check whether d has a strict majority in $(C, V \cup V'_d)$ on the first level, and if so, output $V'' = V'_d$ and halt.

Majority Stage i, $1 < i \leq n$: This stage is entered only if it was not possible to find a successful control action in majority stages $1, \ldots, i-1$. For each candidate $d \in C - \{c\}$, check whether d can tie or beat c up to the ith level via adding at most ℓ voters from V'. To this end, first check whether

$$def^i_{(C,V)}(d) \leq \frac{\ell}{2}; \tag{3.4}$$

$$score^i_{(C,V)}(d) \geq score^i_{(C,V)}(c) - \ell. \tag{3.5}$$

If (3.4) or (3.5) fails to hold, this d is hopeless, so go to the next candidate (or to the next stage if all other candidates have already been checked in this stage). If (3.4) and (3.5) hold, find a set $V'_d \subseteq V'$ of largest cardinality such that $\|V'_d\| \leq \ell$ and all voters in V'_d approve of d up to the ith level but disapprove of c up to the ith level. Check whether

$$score^i_{(C,V\cup V'_d)}(d) \geq score^i_{(C,V\cup V'_d)}(c) \tag{3.6}$$

If (3.6) fails to hold, this d is hopeless, so go to the next candidate (or to the next stage if all other candidates have already been checked in this stage). If (3.6) holds, check whether d has a strict majority in $(C, V \cup V'_d)$ on the ith level, and if so, check whether

$$score^{i-1}_{(C,V\cup V'_d)}(c) \geq maj(V \cup V'_d). \tag{3.7}$$

If (3.7) fails to hold, output $V'' = V'_d$ and halt. Otherwise (i.e., if (3.7) holds), though d might be able to dethrone c by adding V'_d on the ith level, this was not early enough, since c has already won at a previous level. In that case, find a largest set $V'_{cd} \subseteq V'$ such that

1. $\|V'_d \cup V'_{cd}\| \leq \ell$,

2. all voters in V'_{cd} approve of both c and d up to the ith level, and

3. the voters in V'_{cd} are chosen such that c is approved of as late as possible by them (i.e., at levels with a largest possible number, where ties may be broken arbitrarily).

Now, check whether

$$score^{i-1}_{(C,V\cup V'_d\cup V'_{cd})}(c) \geq maj(V\cup V'_d\cup V'_{cd}). \tag{3.8}$$

If (3.8) holds, then this d is hopeless, so go to the next candidate (or to the next stage if all other candidates have already been checked in this stage). Otherwise (i.e., if (3.8) fails to hold), check whether $\|V'_{cd}\| \geq def^{i}_{(C,V\cup V'_d)}(d)$. If so (i.e., d has now a strict majority on level i), output $V'' = V'_d \cup V'_{cd}$ and halt. Note that, by choice of V'_{cd}, (3.6) implies that

$$score^{i}_{(C,V\cup V'_d\cup V'_{cd})}(d) \geq score^{i}_{(C,V\cup V'_d\cup V'_{cd})}(c).$$

Thus, in $(C, V\cup V'_d\cup V'_{cd})$, d ties or beats c and has a strict majority on the ith level (and now, we are sure that d was not too late). Otherwise (i.e., if $\|V'_{cd}\| < def^{i}_{(C,V\cup V'_d)}(d)$), this d is hopeless, so go to the next candidate (or stage).

Approval Stage: This stage is entered only if it was not possible to find a successful control action in majority stages $1,\ldots,n$. First, check if

$$score_{(C,V)}(c) < \left\lfloor \frac{\|V\|+\ell}{2} \right\rfloor + 1. \tag{3.9}$$

If (3.9) fails to hold, output "control impossible" and halt, since we have found no candidate in the majority stages who could tie or beat c and have a strict majority when adding at most ℓ voters from V', so adding any choice of at most ℓ voters from V' would c still leave a strict majority. If (3.9) holds, looping over all candidates $d \in C - \{c\}$, check whether there are $score_{(C,V)}(c) - score_{(C,V)}(d) \leq \ell$ voters in V' who approve of d and disapprove of c. If this is not the case, move on to the next candidate, since d could never catch up on c via adding at most ℓ voters from V'. If it is the case for some $d \in C - \{c\}$, however, add this set of voters (call it V'_d) and check whether

$$score_{(C,V\cup V'_d)}(c) < maj(V\cup V'_d). \tag{3.10}$$

If (3.10) holds, output $V'' = V'_d$ and halt. Otherwise (i.e., if (3.10) fails to hold), check whether

$$\begin{aligned} \ell - \|V'_d\| &\geq \|V'_\emptyset\| \\ &\geq 2\left(score_{(C,V\cup V'_d)}(c) - \frac{\|V\cup V'_d\|}{2}\right), \end{aligned} \tag{3.11}$$

where $V'_\emptyset$ is contained in the set of voters in V' who disapprove of both candidates c and d. If (3.11) does not hold, move on to the next candidate, since after adding these voters c would still have a

strict majority. Otherwise (i.e., if (3.11) holds), add exactly $2\left(score_{(C,V\cup V'_d)}(c) - \|V\cup V'_d\|/2\right)$ voters from $V'_\emptyset$ (denoted by $V'_{\emptyset,+}$). Output $V'' = V'_d \cup V'_{\emptyset,+}$ and halt.

If we have entered the approval stage (because we were not successful in any of the majority stages), but couldn't find any candidate here who was able to dethrone c by adding at most ℓ voters from V', we output "control impossible" and halt.

The correctness of the algorithm follows from the remarks made above. Crucially, note that the algorithm proceeds in the "safest way possible": If there is any successful control action at all then our algorithm finds some successful control action. It is also easy to see that this algorithm runs in polynomial time. (Note that we didn't optimize it in terms of running time; rather, we described it in a way to make it easier to check its correctness.)

The deleting-voters case follows by a similar algorithm (and is omitted here due to space). ❑

Theorem 3.13. *Fallback voting is resistant to constructive control by partition of voters in model TE.*

Proof. Susceptibility holds by Lemma 3.4. To prove NP-hardness, we reduce X3C to our control problem. Let $(B,\mathscr{S})$ be an X3C instance with $B = \{b_1, b_2, \dots, b_{3m}\}$, $m \geq 1$, and a collection $\mathscr{S} = \{S_1, S_2, \dots, S_n\}$ of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$. Our construction, like the corresponding one for SP-AV [ENR09], is based on the corresponding construction for approval voting [HHR07]. Define the election (C,V), where $C = B \cup \{c,x,y,w\} \cup Z$ is the set of candidates, $Z = \{z_1, z_2, \dots, z_n\}$, and w is the distinguished candidate. Define the value $\ell_j = \|\{S_i \in \mathscr{S} \mid b_j \in S_i\}\|$ for each j, $1 \leq j \leq 3m$. Let V consist of the following $4n+2m-1$ voters:

1. For each i, $1 \leq i \leq n$, there is one voter of the form:
$$c \;\; S_i \;\mid\; (B - S_i) \cup \{x,y,w\} \cup Z.$$
2. For each i, $1 \leq i \leq n$, there is one voter of the form:
$$(Z - \{z_i\}) \;\; B_i \;\; w \;\mid\; (B - B_i) \;\; \{c,x,y,z_i\},$$
where $B_i = \{b_j \in B \mid i \leq n - \ell_j\}$.
3. For each i, $1 \leq i \leq n$, there is one voter of the form:
$$c \;\; z_i \;\mid\; B \cup \{x,y,w\} \cup (Z - \{z_i\}).$$
4. There are $n+m$ voters of the form:
$$x \;\mid\; B \cup \{c,y,w\} \cup Z.$$
5. There are $m-1$ voters of the form:
$$y \;\mid\; B \cup \{c,x,w\} \cup Z.$$

Note that for each $i \in \{1,\dots,n\}$ and for each $j \in \{1,\dots,3m\}$, we have $score_{(C,V)}(b_j) = score_{(C,V)}(z_i) = score_{(C,V)}(w) = n$, $score_{(C,V)}(c) = 2n$, $score_{(C,V)}(x) = n+m$, and $score_{(C,V)}(y) = m-1$. Thus, there is no candidate with a strict majority on any level in election (C,V) and, in particular, candidate w is not a unique FV winner.

We claim that $\mathscr{S}$ has an exact cover for B if and only if w can be made a unique FV winner of the resulting election by partition of voters in model TE.

From left to right: Suppose $\mathscr{S}$ contains an exact cover $\mathscr{S}'$ for B. Partition V in the following way. Let V_1 consist of:

- the m voters of the first group that correspond to the exact cover (i.e., those m voters of the form $c \;\; S_i \mid (B-S_i) \cup \{x,y,w\} \cup Z$ for which $S_i \in \mathscr{S}'$),
- the n voters of the third group (who approve of c and z_i), and
- the $n+m$ voters of the fourth group (who approve of only x).

Let $V_2 = V - V_1$. In subelection (C,V_1), no candidate has a strict majority on any level, and c and x tie for first place on the first level, both with score $n+m = \|V_1\|/2$. Thus, there is no candidate proceeding forward to the final round. In subelection (C,V_2), only candidate w has a strict majority, so w is the only participant in the final round of the election and thus has been made a unique FV winner by this partition of voters.

From right to left: Suppose that w can be made a unique FV winner by exerting control by partition of voters in model TE. We can argue for FV as [ENR09] do for SP-AV (see also [HHR07]): Let (V_1,V_2) be such a successful partition. Since we are in model TE, w has to be the unique winner of one of the subelections, say of (C,V_1). Each voter of the form $c \;\; z_i \mid B \cup \{x,y,w\} \cup (Z - \{z_i\})$ has to be in V_2, for otherwise there would be a candidate $z_i \in Z$ with $score_{(C,V_1)}(z_i) = score_{(C,V_1)}(w) = n$, and z_i would get his or her approvals on an earlier level than w. Thus, w would not be the unique winner of subelection (C,V_1). On the other hand, there can be only m voters of the form $c \;\; S_i \mid (B-S_i) \cup \{x,y,w\} \cup Z$ in V_2, for otherwise c would have the highest score in subelection (C,V_2), namely $score_{(C,V_2)}(c) > n+m$, and c would reach this score already on level 1. Thus, c would be the unique winner of subelection (C,V_2), and would also beat w in the run-off because none of the candidates c and w would have a strict majority in election $(\{c,w\},V)$, but c would beat w by approval score. So there are at most m voters of the form $c \;\; S_i \mid (B-S_i) \cup \{x,y,w\} \cup Z$ in V_2. However, there must be exactly m such voters in V_2 and these m voters correspond to an exact cover for B, since otherwise there would be a candidate $b_j \in B$ that has the same score in subelection (C,V_1) as w, namely n points, and b_j would get his or her approvals on an earlier level than w, contradicting the assumption that w is the unique FV winner of subelection (C,V_1). ❑

Finally, we turn to control by partition of voters in model TP. We start with the constructive case.

Theorem 3.14. *Fallback voting is resistant to constructive control by partition of voters in model TP.*

Proof. Susceptibility holds by Lemma 3.4. The proof of resistance is based on the construction of [ENR09, Thm. 3.14]. Let $(B, \mathscr{S})$ be an X3C instance with $B = \{b_1, b_2, \ldots, b_{3m}\}$, $m \geq 1$, and a collection $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ of subsets $S_i \subseteq B$ with $\|S_i\| = 3$ for each i, $1 \leq i \leq n$, and let $n > m+1$. Define the election (C,V), where $C = B \cup F \cup Z \cup \{w,x,y\}$ is the set of candidates, where $F = \{f_1, f_2, \ldots, f_{n+m+1}\}$, $Z = \{z_1, z_2, \ldots, z_n\}$, and w is the distinguished candidate. Define the value $\ell_j = \|\{S_i \in \mathscr{S} \mid b_j \in S_i\}\|$ for each j, $1 \leq j \leq 3m$. Let V consist of the following $6n+2m+2$ voters:

1. For each i, $1 \leq i \leq n$, there is one voter of the form:
$$y \;\; S_i \;\mid\; (B - S_i) \cup F \cup Z \cup \{w,x\}.$$
2. For each i, $1 \leq i \leq n$, there is one voter of the form:
$$y \;\; z_i \;\mid\; B \cup F \cup (Z - \{z_i\}) \cup \{w,x\}.$$
3. For each i, $1 \leq i \leq n$, there is one voter of the form:
$$(Z - \{z_i\}) \;\; B_i \;\; w \;\mid\; (B - B_i) \cup F \cup \{x,y,z_i\},$$
where $B_i = \{b_j \in B \mid i \leq n - \ell_j\}$.
4. There are n voters of the form:
$$Z \;\; B \;\; w \;\mid\; F \cup \{x,y\}.$$
5. There are $n+m+1$ voters of the form:
$$x \;\mid\; B \cup F \cup Z \cup \{w,y\}.$$
6. For each k, $1 \leq k \leq n+m+1$, there is one voter of the form:
$$f_k \;\mid\; B \cup (F - \{f_k\}) \cup Z \cup \{w,x,y\}.$$

The overall scores of the candidates in election (C,V) can be seen in Table 2.

	w	x	y	b_j	f_k	z_i
$score_{(C,V)}$	$2n$	$n+m+1$	$2n$	$2n$	1	$2n$

Table 2: Overall scores in (C,V).

Since the strict majority threshold for V is $3n+m+2$, there is no candidate with a strict majority on any level in election (C,V) and, in particular, since $score_{(C,V)}(y) = score_{(C,V)}(w)$, candidate w is not a unique FV winner.

We claim that $\mathscr{S}$ has an exact cover for B if and only if w can be made a unique FV winner via exerting control by partition of voters in model TP.

From left to right: Suppose $\mathscr{S}$ contains an exact cover $\mathscr{S}'$ for B. Partition V in the following way. Let V_1 consist of:

- the m voters of the first group that correspond to the exact cover (i.e., those m voters of the form $y\ S_i \mid (B - S_i) \cup F \cup Z \cup \{w, x\}$ for which $S_i \in \mathscr{S}'$),
- the n voters of the second group (who approve of y and z_i for all i, $1 \leq i \leq n$), and
- the $n+m+1$ voters of the fifth group (who approve of only x).

Let $V_2 = V - V_1$.

	w	x	y	b_j	f_k	z_i
$score_{(C,V_1)}$	0	$n+m+1$	$n+m$	1	0	1

Table 3: Scores in (C,V_1).

Table 3 shows the overall scores of all candidates in subelection (C,V_1). Since the strict majority threshold is $n+m+1$ in subelection (C,V_1), only candidate x has a strict majority. Thus x is the unique FV winner of subelection (C,V_1) and is proceeding forward to the final round.

	w	x	y	b_j	f_k	z_i
$score_{(C,V_2)}$	$2n$	0	$n-m$	$2n-1$	1	$2n-1$

Table 4: Scores in (C,V_2).

Table 4 shows the overall scores of all candidates in subelection (C,V_2). In this subelection, the strict majority threshold is $2n+1$. As one can see in Table 4, there is no candidate with a strict majority on any level in election (C,V_2). Among all candidates, w has the highest score in subelection (C,V_2) and thus moves forward to the final round.

Since in the final round none of the two candidates, x and w, has a strict majority on any level, and since $score_{(\{w,x\},V)}(w) = 2n > n+m+1 = score_{(\{w,x\},V)}(x)$ because $n > m+1$, candidate w is the unique FV winner in the election resulting from this partition of voters.

From right to left: Suppose that w can be made a unique FV winner by exerting control by partition of voters in model TP. Let $V = (V_1,V_2)$ be some such successful partition. In order to prove that $\mathscr{S}$ has an exact cover for B, we will show the following three statements:

1. The final round of the election corresponding to partition $V = (V_1,V_2)$ consists of the candidates x and w.
2. $\mathscr{S}$ has a cover for B.
3. This cover is an exact cover.

As to (1), among all candidates, only candidate x and the candidates in F have a score less than $score_{(C,V)}(w)$ (see Table 2). Thus, these are the only possible candidates who could face w in the final round.[6] Now, a candidate f_k can get into the final round only if in f_k's first-round subelection

[6]Note that in election (C,V), no candidate has a strict majority. This remains true, if we remove candidates from C. Thus, the winner of the final round is a winner by score and not by majority on any level.

there are only voters from the sixth group and at most one voter from the fifth group.[7] So, if a candidate f_k could make it to the final round then w has to be the unique FV winner of the other first-round subelection. However, this is not possible, since both y and w get $2n$ points in that subelection, and y gains his or her points already on the first level. Thus, no candidate $f_k \in F$ can participate in the final round. It follows that the only way to make w a unique FV winner via exerting control by partition of voters in model TP is that w faces only candidate x in the final round.

As to (2), let x be the unique FV winner of subelection (C,V_1) and w the unique FV winner of subelection (C,V_2). Since in each vote in the third and fourth voter group, each candidate $b_j \in B$ appears on an earlier level than w, there has to be a cover in $\mathscr{S}$ for B, for otherwise there would be at least one $b_j \in B$ who ties w in (C,V_2) by score, and reaches that score on an earlier level than w. Let the size of the cover be m'. Note that $m' \geq m$.

As to (3), note that each voter of the second group has to be in (C,V_1), for otherwise there would be at least one $z_i \in Z$ who ties w in (C,V_2) by score, and reaches that score on an earlier level than w. The following must hold:

$$score_{(C,V_1)}(x) - score_{(C,V_1)}(y) = (n+m+1) - (n+m') = m+1-m' > 0.$$

This is possible only if $m' = m$. Hence, $\mathscr{S}$ has an *exact* cover for B, which completes the proof. ❑

The following construction will be used to handle the destructive case of control by partition of voters in model TP.

Construction 3.15. *Let $(B,\mathscr{S},k)$ be a given instance of* Hitting Set, *where $B = \{b_1, b_2, \ldots, b_m\}$ is a set, $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ is a collection of nonempty subsets $S_i \subseteq B$, and $k < m$ is a positive integer.*[8]

Define the election (C,V), where $C = B \cup D \cup E \cup \{c,w\}$ is the candidate set with $D = \{d_1, \ldots, d_{2(m+1)}\}$ and $E = \{e_1, \ldots, e_{2(m-1)}\}$ and where V consists of the following $2n(k+1)+4m+2mk$ voters:

1. *For each i, $1 \leq i \leq n$, there are $k+1$ voters of the form:*

$$w \;\; S_i \;\; c \;\mid\; D \cup E \cup (B - S_i).$$

2. *For each j, $1 \leq j \leq m$, there is one voter of the form:*

$$c \;\; b_j \;\; w \;\mid\; (B - \{b_j\}) \cup D \cup E.$$

3. *For each j, $1 \leq j \leq m$, there are $(k-1)$ voters of the form:*

$$b_j \;\mid\; (B - \{b_j\}) \cup D \cup E \cup \{c,w\}.$$

[7] Otherwise, since $score_{(C,V)}(f_k) = 1$, it might happen that y or a candidate $z_i \in Z$ would also be a winner in this subelection and would move forward to the final round.

[8] Note that the assumption $k < m$ can be made without loss of generality, since the problem Hitting Set becomes trivial if $k = m$.

	c	w	b_j	d_p	e_r
score[1]	$n(k+1)+2m+mk$	$n(k+1)+1$	$k-1$	≤ 1	1
score[2]	$n(k+1)+2m+mk+1$	$n(k+1)+mk+k$	$\leq k+n(k+1)$	1	1
score[3]	$\geq n(k+1)+2m+mk+1$	$n(k+1)+mk+k+2m+1$	$\leq k+n(k+1)$	1	1

Table 5: Scores in (C,V).

4. *For each p, $1 \leq p \leq m+1$, there is one voter of the form:*

$$d_{2(p-1)+1} \;\; d_{2p} \;\; w \;\mid\; B \cup (D - \{d_{2(p-1)+1}, d_{2p}\}) \cup E \cup \{c\}.$$

5. *For each r, $1 \leq r \leq 2(m-1)$, there is one voter of the form:*

$$e_r \;\mid\; B \cup D \cup (E - \{e_r\} \cup \{c,w\}.$$

6. *There are $n(k+1)+m-k+1$ voters of the form:*

$$c \;\mid\; B \cup D \cup E \cup \{w\}.$$

7. *There are $mk+k-1$ voters of the form:*

$$c \;\; w \;\mid\; B \cup D \cup E.$$

8. *There is one voter of the form:*

$$w \;\; c \;\mid\; B \cup D \cup E.$$

The strict majority threshold for V is $\mathrm{maj}(V) = n(k+1)+2m+mk+1$. In election (C,V), only the two candidates c and w reach a strict majority, w on the third level and c on the second level (see Table 5). Thus c is the unique level 2 FV winner of election (C,V).

The proof of Theorem 3.17 will make use of the following claim.

Claim 3.16. *In election (C,V) from Construction 3.15, for every partition of V into V_1 and V_2, candidate c is an FV winner of either (C,V_1) or (C,V_2).*

Proof. For a contradiction, suppose that in both subelections, (C,V_1) and (C,V_2), candidate c is not an FV winner. In particular, c can have no strict majority in either of (C,V_1) and (C,V_2). Since $score^1_{(C,V)}(c) = \|V\|/2$, the two subelections must satisfy the following conditions:

1. Both $\|V_1\|$ and $\|V_2\|$ are even numbers and
2. $score^1_{(C,V_1)}(c) = \|V_1\|/2$ and $score^1_{(C,V_2)}(c) = \|V_2\|/2$.

Otherwise, c would have a strict majority already on the first level in one of the subelections. Since in both subelections c has only one point less than the strict majority threshold already on the first level, and since c will get a strict majority no later than on the second level, in both subelections

there must be candidates whose level 2 scores are higher than the level 2 score of candidate c. Table 5 shows the level 2 scores of all candidates. Only candidates w and a $b_j \in B$ have a chance to tie or beat candidate c on that level.

Essentially, there are two possibilities for winning the two subelections. First, it is possible that both subelections are won by two distinct candidates from B (say, b_x is a winner of (C,V_1) and b_y is a winner of (C,V_2)). Thus the following must hold:

$$\begin{array}{rcl} score^2_{(C,V_1)}(b_x) + score^2_{(C,V_2)}(b_y) & \geq & score^2_{(C,V)}(c) \\ 2n(k+1)+2k-n(k+1) & \geq & n(k+1)+mk+2m+1 \\ 2k & \geq & mk+2m+1 \\ 0 & \geq & (m-2)k+2m+1. \end{array}$$

This is a contradiction to the basic assumption that both $k > 0$ and $m > 0$. Thus only the second possibility for c to lose both subelections remains, namely that one subelection, say (C,V_1), is won by a candidate from B, say b_x, and the other subelection, (C,V_2), is won by candidate w. Then it must hold that:

$$\begin{array}{rcl} score^2_{(C,V_1)}(b_x) + score^2_{(C,V_2)}(w) & \geq & score^2_{(C,V)}(c) \\ n(k+1)+k-n(k+1)+n(k+1)+mk+k & \geq & n(k+1)+mk+2m+1 \\ 2k & \geq & 2m+1. \end{array}$$

This is a contradiction to the assumption that $k < m$, so c must be an FV winner in one of the subelections. ❑

Theorem 3.17. *Fallback voting is resistant to destructive control by partition of voters in model TP.*

Proof. Susceptibility holds by Lemma 3.4. To prove NP-hardness, we reduce Hitting Set to our control problem. Consider the election (C,V) constructed according to Construction 3.15 from a given Hitting Set instance $(B, \mathscr{S}, k)$, where $B = \{b_1, \dots, b_m\}$ is a set, $\mathscr{S} = \{S_1, \dots, S_n\}$ is a collection of nonempty subsets $S_i \subseteq B$, and $k < m$ is a positive integer.

We claim that $\mathscr{S}$ has a hitting set $B' \subseteq B$ of size k if and only if c can be prevented from winning by partition of voters in model TP.

From left to right: Suppose, $B' \subseteq B$ is a hitting set of size k for $\mathscr{S}$. Partition V into V_1 and V_2 the following way. Let V_1 consist of those voters of the second group where $b_j \in B'$ and of those voters of the third group where $b_j \in B'$. Let $V_2 = V - V_1$. In (C,V_1), no candidate reaches a strict majority (where $\mathrm{maj}(V_1) = \|B'\| + 1 = k+1$), and candidates c, w, and $b_j \in B'$ win the election with an approval score of k (see Table 6).

The scores in election (C,V_2) are shown in Table 7.

Since in election (C,V_2) no candidate from B wins, the candidates participating in the final round are $B' \cup \{c,w\}$. The scores in the final election $(B' \cup \{c,w\}, V)$ can be seen in Table 8. Since candidates c and w are both level 2 FV winners, candidate c is no longer the unique FV winner of the election.

	c	w	$b_j \in B'$	$b_j \notin B'$
score[1]	k	0	$k-1$	0
score[2]	k	0	k	0
score[3]	k	k	k	0

Table 6: Scores in (C,V_1).

	c	w	$b_j \notin B'$	$b_j \in B'$
score[1]	$n(k+1)+2m-k+mk$	$n(k+1)+1$	$k-1$	0
score[2]	$n(k+1)+2m-k+mk+1$	$n(k+1)+mk+k$	$\leq k+n(k+1)$	$\leq n(k+1)$
score[3]	$\geq n(k+1)+2m-k+mk+1$	$n(k+1)+mk+2m+1$	$\leq k+n(k+1)$	$\leq n(k+1)$

Table 7: Scores in (C,V_2).

	c	w	$b_j \in B'$
score[1]	$n(k+1)+2m+mk$	$n(k+1)+m+2$	$k-1$
score[2]	$n(k+1)+2m+mk+1$	$n(k+1)+2m+mk+1$	$\leq k+n(k+1)$

Table 8: Scores in $(B'\cup\{c,w\},V)$.

From right to left: Suppose candidate c can be prevented from winnning by partition of voters in model TP. From Claim 3.16 it follows that candidate c participates in the final round. For a contradiction, suppose that $\mathscr{S}$ has not a hitting set of size k. Since c has a strict majority of approvals, c has to be tied with or lose against another candidate by a strict majority at some level. Only candidate w has a strict majority of approvals, so w has to tie or beat c at some level in the final round. Because of the scores of the candidates from D and E we may assume that only candidates from B are participating in the final round besides c and w. Let $B' \subseteq B$ be the set of candidates who also participate in the final round and let ℓ be the number of sets in $\mathscr{S}$ not hit by B'. Note that w cannot reach a strict majority of approvals on the first level, so we consider the level 2 scores of c and w:

$$\begin{aligned} score^2_{(B'\cup\{c,w\},V)}(c) &= n(k+1)+2m+mk+1+\ell(k+1) \quad \text{and} \\ score^2_{(B'\cup\{c,w\},V)}(w) &= n(k+1)+2m+mk+k-\|B'\|+1. \end{aligned}$$

Since c has a strict majority already on the second level, w must tie or beat c on this level, so the following must hold:

$$\begin{aligned} score^2_{(B'\cup\{c,w\},V)}(c) - score^2_{(B'\cup\{c,w\},V)}(w) &\leq 0 \\ n(k+1)+2m+mk+1+\ell(k+1)-n(k+1)-2m-mk-k+\|B'\|-1 &\leq 0 \\ \|B'\|-k+\ell(k+1) &\leq 0. \end{aligned}$$

This is possible only if $\ell=0$, which contradicts to our assumption that there are sets in $\mathscr{S}$ that are not hit by B'. From $\ell=0$ it follows that $\|B'\|\leq k$, so $\mathscr{S}$ has a hitting set of size at most k. ❑

4 Conclusions and Open Questions

We have shown that Brams and Sanver's fallback voting system [BS09] is, like plurality voting and SP-AV, fully resistant to candidate control. Also, like Copeland voting [FHHR09a] and SP-AV, fallback voting is fully resistant to constructive control. Regarding voter control, all eight cases in FV are susceptible, and we have shown resistance to constructive control by adding, by deleting, and by partition of voters in models TE and TP, and by destructive control to partition of voters in model TP. We have also shown vulnerability to destructive control by adding and by deleting voters. Only one case remains open: destructive control by partition of voters in model TE. It would be interesting to know whether FV is resistant or vulnerable to this control type.

Plurality voting is one of the other two systems for which full resistance to candidate control is known [HHR07], but it has fewer resistances to voter control than fallback voting. SP-AV (the other system with known full resistance to candidate control) does have the same number of proven resistances [ENR09] to voter control as fallback voting. However, as has been argued in the introduction, it is less natural a system than fallback voting. Also, it is still possible that fallback voting might turn out to have even one more resistance to control than SP-AV in total.

Of course, resistance to control is not the only—and probably not even the most important—criterion to guide one's choice of voting system. Many other properties of voting systems (especially their social choice weaknesses and strengths) are important as well and perhaps even more important. For example, representing votes in plurality is a slightly simpler task than in fallback voting or SP-AV: Plurality voters simply give a ranking of the candidates and the candidates with the most top positions win, whereas fallback and SP-AV voters provide both their approvals/disapprovals of the candidates and a ranking of the candidates (of all candidates in SP-AV and of only the approved candidates in fallback voting). Also, winner determination in fallback voting and in SP-AV is a slightly more complicated task than in plurality voting—though still easy. Regarding the social choice benefits of FV, we mention that it satisfies, e.g., monotonicity and refer to [BS09] for a more detailed discussion and further interesting results.

Supposing one does care about control resistance, when choosing a voting system one's choice will most likely (along with the system's social choice properties, of course) depend on the types of control one cares most about in the intended application. Also, when comparing voting systems, one should weigh the nine immunities, four resistances, and nine vulnerabilities to control approval voting is known to possess [HHR07] against FV's at least 19 (and possibly even 20) resistances and at least two (and at most three) vulnerabilities to control.

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A Some Results of [HHR07] Used in Section 3.2

Theorem A.1 ([HHR07])**.**

1. *If a voiced voting system is susceptible to destructive control by partition of voters (in model TE or TP), then it is susceptible to destructive control by deleting voters.*
2. *Each voiced voting system is susceptible to constructive control by deleting candidates.*
3. *Each voiced voting system is susceptible to destructive control by adding candidates.*[9]

Theorem A.2 ([HHR07])**.**

1. *A voting system is susceptible to constructive control by adding candidates if and only if it is susceptible to destructive control by deleting candidates.*
2. *A voting system is susceptible to constructive control by deleting candidates if and only if it is susceptible to destructive control by adding candidates.*

[9]Following Bartholdi et al. [BTT92], Hemaspaandra et al. [HHR07] considered only the case of control by adding a limited number of candidates—the “unlimited” case was introduced only in (the conference precursors of) [FHHR09a]. However, it is easy to see that all results about control by adding candidates stated in Theorems A.1, A.2, and A.3 hold true in both the limited and the unlimited case.

3. *A voting system is susceptible to constructive control by adding voters if and only if it is susceptible to destructive control by deleting voters.*

4. *A voting system is susceptible to constructive control by deleting voters if and only if it is susceptible to destructive control by adding voters.*

Theorem A.3 ([HHR07]**).** 1. *If a voting system is susceptible to constructive control by partition of voters (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*

2. *If a voting system is susceptible to constructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to constructive control by deleting candidates.*

3. *If a voting system is susceptible to constructive control by partition of voters in model TE, then it is susceptible to constructive control by deleting voters.*

4. *If a voting system is susceptible to destructive control by partition or run-off partition of candidates (in model TE or TP), then it is susceptible to destructive control by deleting candidates.*